Complex Geometry Exercises

Week 11

Exercise 1. *Prove that the standard identity*

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$

extends to vector-valued 1*-forms* ϕ*:*

$$d\phi(X,Y) = [X,\phi(Y)] - [Y,\phi(X)] - \phi([X,Y]) \ .$$

Exercise 2. *Consider* $H_\lambda = \mathbb{C}^n \setminus \{0\}/\langle\lambda\rangle$ *the Hopf surface, with* $\lambda \in \mathbb{C}^*$, $|\lambda| < 1$, *for* $n \geq 2$.

(i) Show that $f \in \mathrm{Aut}(H_\lambda)$ *extends to a biholomorphism* $\widetilde{f} : \mathbb{C}^n \to \mathbb{C}^n$ *with* $\widetilde{f}(0) = 0$.

(ii) Show that $\widetilde{f}(\lambda z) = \lambda \widetilde{f}(z)$ *(as opposed to* $\lambda \widetilde{f}(\lambda z) = \widetilde{f}(z)$*).*

(iii) Prove that $\partial \widetilde{f}(\lambda^n z) = \partial \widetilde{f}(z)$ *for all* $n \in \mathbb{Z}$, *so* $\widetilde{f} \in \mathrm{GL}(n,\mathbb{C})$.

(iv) Compute $\mathrm{Aut}(H_\lambda)$.

Exercise 3. *Let* X *be a Kähler manifold with* $c_1(K_X) < 0$. *Prove that* $H^2(X, \tau_X) = 0$.

Exercise 4. *Consider a flat torus* $\mathbb{T}^n = \mathbb{C}^n/\Gamma$.

(i) Find an explicit basis of $\mathcal{H}^{p,q}(\mathbb{T}^n)$. *Show that* $h^{p,q} = \binom{n}{p}\binom{n}{q}$

(ii) Compute the virtual dimension of complex strucutres $\mathcal{M}_{\mathbb{T}^n}$.

(iii) Show that the obstruction map $H^1(\mathbb{T}^n, \tau_{\mathbb{T}^n}) \xrightarrow{\Phi} H^2(\mathbb{T}^n, \tau_{\mathbb{T}^n})$ *vanishes.*

(iv) Study how the group $\mathrm{Aut}^0(\mathbb{T}^n)$ *acts on* $H^1(\mathbb{T}^n, \tau_{\mathbb{T}^n})$.

(v) Compute the dimension of $\mathcal{M}^0_{\mathbb{T}^n} \coloneqq \Psi^{-1}(0)/\mathrm{Aut}^0(\mathbb{T}^n)$.

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Exercise 5 (Moduli space of complex tori)**.**

(i) *Prove that a complex torus of dimension n is determined by a map $\mathbb{Z}^{2n} \to \mathbb{C}^n \cong \mathbb{R}^{2n}$, and thus, can be identified with an element of* $\mathrm{GL}(2n, \mathbb{R})$.

So there is a surjective map $\mathrm{GL}(2n, \mathbb{R}) \to \mathcal{M}_{\mathbb{T}^n}$.

(ii) *Prove the actions of* $\mathrm{GL}(n, \mathbb{C})$ *and* $\mathrm{GL}(2n, \mathbb{Z})$ *on* $\mathrm{GL}(2n, \mathbb{R})$ *induce the trivial action on* $\mathcal{M}_{\mathbb{T}^n}$.

(iii) *Conclude that* $\mathrm{GL}(n, \mathbb{C}) \setminus \mathrm{GL}(2n, \mathbb{R})$ *is a "covering" space of* $\mathcal{M}_{\mathbb{T}^n}$.

(iv) *Prove that* $\mathcal{M}_{\mathbb{T}^n}$ *is isomorphic to the bi-quotient*

$$\mathrm{GL}(n, \mathbb{C}) \setminus \mathrm{GL}(2n, \mathbb{R}) / \mathrm{GL}(2n, \mathbb{Z}) .$$

(v) *Prove that* $\mathrm{GL}(2n, \mathbb{Z})$ *does not act properly discontinuous on* $\mathrm{GL}(n, \mathbb{C}) \setminus \mathrm{GL}(2n, \mathbb{R})$ *for* $n \geq 2$, *so* $\mathcal{M}_{\mathbb{T}^n}$ *is not Hausdorff.*

(vi) *For* $n = 1$, *prove that*

$$\mathrm{GL}(1, \mathbb{C}) \setminus \mathrm{GL}(2, \mathbb{R}) \cong \mathbb{H}^2 \sqcup \mathbb{H}^2 ,$$

Where $\mathbb{H}^2 = \{x + iy \in \mathbb{C} \mid y > 0\}$ *is the hyperbolic plane, and the two copies are distinguished by orientation.*

(vii) *Give a geometric description of* $\mathcal{M}_{\mathbb{T}^1}$.

(viii) *Show that* $\mathcal{M}_{\mathbb{T}^1}$ *is not compact. Can you find a natural compactification?*

(ix) *Show that* $\mathcal{M}_{\mathbb{T}^1}$ *is not a manifold, but an orbifold. Can you compute the stabiliser of the non-smooth points?*

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Exercise 6 (Moduli space of principally polarised tori). *From Exercise 6 Sheet 9, we know that a complex torus $\mathbb{C}^n/\Gamma$ is projective iff ω restricts to a symplectic form $\Gamma \times \Gamma \to \mathbb{Z}$. Fix a symplectic structure ω (equivalently a hermitian metric) on $\mathbb{C}^n$.*

(i) Prove that a projective torus of dimension n is determined by a symplectic map $\mathbb{Z}^{2n} \to \mathbb{C}^n \cong \mathbb{R}^{2n}$, and thus, can be identified with an element of $\mathrm{Sp}(2n, \mathbb{R})$.

We define $\mathcal{M}^n_{Ab}$ the moduli of principally polarised n-dimensional tori to be the collection of tori given by the construction in (i), modulo automorphism.

(ii) Show the actions of $U(n)$ and $\mathrm{Sp}(2n, \mathbb{Z})$ on $\mathrm{Sp}(2n, \mathbb{R})$ induce the trivial action on $\mathcal{M}^n_{Ab}$.

(iii) Prove that $\mathcal{M}^n_{Ab}$ is isomorphic to the bi-quotient

$$\mathrm{U}(n) \setminus \mathrm{Sp}(2n, \mathbb{R}) / \mathrm{Sp}(2n, \mathbb{Z}) .$$

(iv) Check that, for $n = 1$, we have $\mathcal{M}^1_{Ab} \cong \mathcal{M}_{\mathbb{T}^1}$.

(v) Prove that $\mathrm{Sp}(2n, \mathbb{Z})$ acts properly discontinuously on $\mathrm{U}(n) \setminus \mathrm{Sp}(2n, \mathbb{R})$. In particular, the moduli space $\mathcal{M}^n_{Ab}$ is Hausdorff for all $n \geq 2$.